INTERACTION BETWEEN MATHEMATICS AND PHYSICS

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RESUMEN: *Actualmente existe una importante interfaz entre matemáticas y física teórica, que ha producido áreas completamente nuevas. Este artículo está basado en un debate en una mesa redonda organizada en el entorno del International Congress of Mathematicians en 2006 de Madrid, explora algunos de estos temas: los diferentes objetivos y pasado de ambas disciplinas, las interacciones actuales y sus precedentes, las posibilidades para el futuro y el papel de las matemáticas para entender el mundo en que vivimos.*

PALABRAS CLAVE: *Mathematical Physics, Theoretical Physics, Mathematics.*

ABSTRACT: There is at the moment a highly active interface between mathematics and theoretical physics, which extends into completely new areas of both disciplines. This article, based on a round table discussion which took place as part of the activities around the 2006 International Congress of Mathematicians in Madrid, explores some of the issues involved: the differing goals and backgrounds of the two communities, today's interactions and their precedents, the possibilities for the future and the role of mathematics itself in understanding the world in which we live.

KEY WORDS: Mathematical Physics, Theoretical Physics, Mathematics.

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Mathematicians and physicists

According to Galileo Galilei, "Mathematics is the language with which God has written the universe," a view echoed 400 years later in Eugene Wigner's paper entitled "The Unreasonable Effectiveness of Mathematics in the Natural Sciences". The past 30 years has seen a significant change, however, which some have characterized as "The Unreasonable Effectiveness of Physics in Mathematics". How has this come about? Is it good for mathematics? Is it good for physics?

To understand the current situation one needs also to understand the differing aims and methods of both groups. A physicist's attempt to understand physical reality is based on experiments, measurements and the recognition and formulation of laws. To frame those rules, mathematics is necessary, but however sophisticated a tool, it is used for the purpose of better understanding the physical processes. Its ultimate validation is its agreement with experimentation, when that is possible. Thus physicists believe in quantum field theory not because it is a rigorous piece of mathematics, but because it gives them the correct answers to many decimal places.

They work in different ways from mathematicians, attacking current problems with a huge concentration of forces. They have no time to wait for the full mathematical theory but proceed with great momentum that carries them beyond the stage where the hypotheses are testable.

Contrast this with the mathematician, willing to wait years to complete a theory, like Andrew Wiles's celebrated proof of Fermat's theorem or, closer to physical reality, Carl Friedrich Gauss's 25 years of secretly studying the differential geometry of surfaces (the physical reality that Gauss was attempting to describe there was founded in geodesy). The pure mathematician is, in the public's view, a practitioner of an art which "possesses not only truth, but supreme beauty –a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature", in Bertrand Russell's words. Can there be any common ground for both communities to work in?

One answer is to say that science is not actually describing physical reality but is concerned with human understanding of it. In this view, the beauty and elegance of mathematics is a guide towards a theory that has a coherence and simplicity that aids our comprehension of nature. But beauty alone can lead the physicist astray. Who can deny that Johannes Kepler's original view of the solar system based on the Platonic solids was beautiful? But it was plain wrong. Kepler tried hard to avoid the conclusion that the planetary orbits were elliptic but in the end he had to admit it was a fact. Whether he appreciated it as a manifestation of another beautiful piece of mathematics is not clear but what actually happened was that one elegant model was replaced by another more sophisticated one.

It is perfectly possible to change one's view of what constitutes a simple elegant theory. The cause need not even be a failure of the theory to agree with experiment. It can also come from a better understanding of mathematics. Albert Einstein in his younger days complained "since the mathematicians have invaded the theory of relativity, I do not understand it myself" but 20 years later offered the opinion that "nature is the realization of the simplest possible mathematical ideas."

But mathematicians rarely pursue art for art's sake. They are always out to discover and understand. Here is Galileo again: "All truths are easy to understand once they are discovered; the point is to discover them." And for mathematicians knowing what is true, or discovering what is true, is a matter of analogy and metaphor, comparisons with other parts of mathematics –or more frequently from outside mathematics, in the physical sciences. Mathematical proof is often another question involving technique, knowledge of what others have done and sheer invention.

Nor is mathematics a static discipline. It has its own internal dynamics: some fields develop and brush against neighbouring areas, some settle down to steady progress for a few decades and then explode. Some of the growth areas of the 1960s, for example, when resources were poured into science, became quiescent twenty years later but then sprang back into life. Or to take a longer-term view one might pick out Bernhard Riemann's work in the mid 19th century on differential geometry; its subsequent development in higher dimensions by Gregorio Ricci-Curbastro in 1904 prepared it for its phenomenal expansion when it was seen as the language in which to express Einstein's general relativity. More recently, and certainly at the International Congress in Madrid, one experienced the shift from deterministic to stochastic methods, which have their origins in the 19th century physicists' study of thermodynamics. These movements sometimes originate from developments within the subject, sometimes from external influences.

Both communities of mathematicians and physicists are alive and evolving, and aiming at discovery, but their backgrounds and motivations differ. This diversity is a source of strength if the two groups can focus on a common problem.

Current interactions

Perhaps the most exciting interaction between physics and mathematics at the moment is in Quantum Field Theory and String Theory. Any interaction is a two-way process but in the past few years it is the predictive power of String Theory in *pure* mathematics that is the most astonishing feature. New facts and coincidences are being pointed out, not only in traditional areas with a common interface, but also way beyond that, in algebraic geometry and number theory. It is as if today's theoretical physics has had the power to jump into the interior of pure mathematics and tear it apart.

The cynical might say that this is not an achievement of String Theory but a manifestation of its failure to be predictive about actual physical reality. Is it really a physical theory, or simply a set of analogies? Are mathematicians just feeding off the physicists' intuition because pure mathematics is the only place where the theory is applicable? The counterargument is to assert that String Theory is a consistent theory but it is so complicated that it has to use every tool in the mathematician's cupboard. It may still be true that "nature is the realization of the simplest possible mathematical ideas", it's just that you need to know a lot of mathematics to see how simple it is.

String theorists would freely admit that they don't know what the theory is, but they are fairly sure that what they have is a genuine theory. What they observe is its implications at different limits of coupling constants where it makes contact with other areas of mathematics. The fundamental concepts in the *terra incognita* at its centre are unknown yet its deep consistency unearths structures across a wide range of mathematics. They also admit that is harder than they thought when the possibilities opened up in the mid 1980s, but by being harder it has drawn them closer to mathematics and they are quite happy to use the predictive power within that domain, given that the physical experiments are currently impractical.

Most mathematicians welcome this interaction and are happy to use the "unreasonable effectiveness of the equations of mathematical physics in pure mathematics". These have a history, since before String Theory. Hermann Weyl investigated the representation theory of groups because of its use in quantum mechanics, but it is now a tool throughout mathematics: in algebra, geometry and number theory. The more recent interactions involve the Fields Medal-winning work of Simon Donaldson using Yang-Mills equations to probe the topology of four-dimensional manifolds and that of Vaughan Jones and Edward Witten in defining knot invariants. These are practical (in a mathematical sense) theories that can be put to use in many areas, but often the crucial advances were achieved by pursuing the physicist's intuition. Some would say that these are advances that perhaps mathematicians did not deserve.

In the current phase of interaction, mathematicians are now becoming familiar with the physicists' way of wrapping up mathematical information in a partition function. This becomes a formal means of counting objects that have been considered individually in the past but not systematically in such a way. These objects might be algebraic curves, or numbers of intersections or numbers of solutions to certain equations, all wrapped up in a generating function. Sometimes there is a subtlety in counting multiplicities which has eluded the mathematicians but which is natural for the physicists and leads to functional equations which the generating functions satisfy.

From the physicist's point of view the process of interaction works like this:

(i) Start with a mathematical problem.
(ii) Formulate it as a (non-rigorous) field theory.
(iii) Study different pictures and limits.
(iv) Discover new mathematical objects.

Given this global view, one can highlight certain examples. Donaldson theory is viewed as perturbative. The distinct, non-perturbative picture of the same theory yields Seiberg-Witten theory. Here was a piece of mathematics which, we were told, would describe in a different way the same invariants as Donaldson did. And why? Because they are two limiting forms of the same quantum field theory.

What the mathematicians found was that it gave them a brand-new method to prove efficiently precisely the results they were finding it hard to achieve with standard Donaldson theory. There was a period in 1995 when geometers burnt the midnight oil to race each other to proofs of some longstanding conjectures using this new method. "Donaldson theory is dead" they would say, but in the fullness of time it became clear that the two methods were in fact complementary. A mathematical proof of the link between the two is still not quite achieved ten years later –this is testament to the power of the physics to unearth the truth.

The point here is that the perturbative/non-perturbative physics view created a double-edged sword for mathematicians to attack some old problems. In another setting, that of Chern-Simons theory applied to the theory of knots and links, the perturbative view gives the Vassiliev invariants and the non-perturbative the Jones-Witten polynomial invariants.

It may have been that some of these theories were developed independent of this view, but their universality when seen from this perspective not only gives them a hidden cohesion, but in the long term the experience with mathematics may be used in reverse to provide input into the efforts to construct a rigorous mathematical framework for Quantum Field Theory and String Theory.

Precedents

Is this interaction with physics new? Are there historical precedents? It is difficult to transport oneself into the past or to guess what the thoughts and stimuli of mathematicians were then. The numbers were fewer, scientific activity as a whole was on a smaller scale. A physicist or mathematician in the past was not so specialized and was thus open through correspondence or discussions with scientific colleagues to a host of inputs across a broad spectrum of science. Yet usually the flow of information was from mathematics to physics. It was rare for a subject as pure as geometry to feel the effects.

There are nevertheless instances of this happening. In the mid 19th century Riemann introduced analytical methods into the algebraic geometry of curves. These were sometimes "proved" by appeal to physical principles such as the Dirichlet principle, a technique motivated by the physical tenet that nature works by minimizing actions and energy. Yet the whole apparatus of differentials and theta functions enabled remarkable results to be proved or rendered obvious; special facts like the existence of precisely 28 bitangents to a quartic curve or 120 tritangent planes to a genus four curve are not so far removed in spirit from the remarkable count of rational curves on the quintic threefold by Candelas *et al.* which is the most startling application of the string theorists' mirror symmetry in algebraic geometry. If one looks at the journals of the time, one will also see a very rapid succession of applications of these methods before a settling down at the end of the century to a mixture of techniques.

If one goes further back in time, then giants like Newton stand out of course in fusing mathematics with physics. But this long-past world really was a foreign country, as anyone who has pored over Newton's notebooks will see, where detailed calculations fill the gaps in long discussions of biblical history.

The present is different because it consists of a refocusing of previously divergent paths in mathematics. The rapid expansion in mathematics in the post-war period was channelled into new areas where there were plenty of problems to solve. But that was not necessarily sustained –the problems became more challenging and new techniques were more difficult to find. Caught up in their own world, mathematicians were less open to ideas from physics. Perhaps a good demonstration of this, and how it was overcome, is the index theorem, one of the most important results in the 20th century for unifying different branches of mathematics.

In 1962 Michael Atiyah and Isadore Singer began work on this theorem, for which they were awarded the Abel Prize in 2004. It began as a quest to explain why certain rational numbers in algebraic topology are integers –were they related to dimensions of vector spaces? In pursuing this aim, they rediscovered one of the fundamental differential operators of physics– the Dirac operator. Of course this was not in quite the same setting –they were working in Riemannian geometry rather than Einstein's space-time, but it was essentially the same operator. There began several proofs: the first two used ideas from two of the most active areas of mathematics at the time. The first was a part of algebraic topology –Rene Thom's cobordism theory. Then came the second proof (with a wider range of applicability) using the far-reaching abstract ideas of Alexandre Grothendieck in algebraic geometry. Much later, in the mid 1970s, a third proof involving the heat kernel and differential geometry emerged.

Yet at the same time physicists were in the process of rediscovering the theorem. Singer once remarked that this was taking place in the adjacent corridor at MIT to his own office. For the physicists, who were studying what they called anomalies, the heat kernel expansions were commonplace. The new ideas for them were the links with algebraic topology. So the evolution of their theorem was proceeding in the opposite direction and only in the late 1970s, as both mathematicians and physicists got interested in the Yang-Mills equations, did they really put their heads together. This was a crucial moment, when the mathematicians realized that physicists had uncovered a completely new way of looking at what they called connections and physicists realized that the problems that had been bothering them for some time could be resolved by the use of some quite sophisticated mathematics which was only then being developed. It was no longer true that the only mathematics a physicist needed to know was how to integrate by parts!

In many respects this was an influx of "classical ideas" from physics to mathematics but it was not long before

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those mathematical results could be viewed as part of a much bigger quantum field theory and the full force of the physicists' intuition could be brought into play.

There is, then, a difference between the current interactions and those of previous periods. It involves the scale of interactions, the range of mathematics being utilized and the changing dynamics of the subject. And still the underlying irony is that the mathematical results that are being correctly predicted are often based on a nonrigorously posed quantum field theory.

Why physics?

Misha Gromov, the celebrated mathematician, winner of the Wolf, Kyoto and Balzan Prizes has recently been giving seminars entitled "How a Mathematician May Think of Proteins". His recent researches, carried out at the Institut des Hautes Etudes Scientifiques in Paris, or the Courant Institute in New York, have concentrated on many problems in biology. Does this herald a new set of intuitions entering into the mathematical mainstream, a new area of science where mathematics can benefit from outside influences? Why should physics be the only partner to this most abstract of sciences?

The interaction with biology is at an early phase but it seems unlikely that it will have as much influence as physics in the development of new mathematical ideas. We discussed earlier the cultural differences and working practices of the two communities of mathematicians and physicists. These are far more pronounced when one considers biologists. The first is the size –there are perhaps a million research workers in biology, and in the region of 60,000 in mathematics. The biologists work in larger groups and much of their activity is experimental, with a wide diversity of experiments. Actually checking the validity of the experiments is very difficult let alone formulating possible laws to explain the results. And the pace of advance is probably faster than in theoretical physics. If a mathematician produces a paper once a year, a biologist will do so once a month.

It is equally true that the subjects that most directly impinge on biology such as biochemistry are sciences which are furthest away from mathematics. The closest discipline to mathematics is, and always will be, physics. Biology is growing and growing fast, not only in research but also as a popular subject in schools and university, but its enormous achievements have largely been carried out without theoretical mathematical input. The same could not be said of physics.

Nevertheless, just because String Theory has an important link with mathematics at the moment it does not mean that this should be an exclusive interface. There is surely room for absorbing not just new problems but new points of view from other areas such as condensed matter physics.

Why mathematics?

String theory has come in for criticism recently, with heightened public awareness coming from the publication of books such as Peter Woit's "Not Even Wrong" or Lee Smolin's "The Trouble with Physics". The suspicion is that, without experimental evidence, String Theory has become too close a friend of pure mathematics and has strayed too far from what physicists should be doing. Implicit in this is the criticism that what pure mathematicians do is so disconnected from the real world that it can be of no use. Why, for example, should the government of a developing country put resources into either mathematics or theoretical physics?

G. H. Hardy is usually quoted as being proud, in the period after the First World War, of the uselessness, or harmlessness, of pure mathematics, but he also said "Pure mathematics is on the whole distinctly more useful than applied. For what is useful above all is technique, and mathematical technique is taught mainly through pure mathematics." Nehru also had a high opinion of mathematics " Mathematics is the vehicle of exact scientific thought. It has widened the horizons of the human mind tremendously". He certainly valued the contribution to a developing country of theoretical science and mathematics.

Working mathematicians always feel that there is some link from their world to recognizable reality. How else can they discover things? Hardy writes: "I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply the notes of our observations." Some chain of reasoning, analogy or technique links what we do with physical reality. If we tug on that chain then, however small the impact, we hope that our perception of reality is enhanced.

Science is full of dead ends, theories that came to nothing, but a mathematical proof is always valid. One hopes it can contribute in some way to basic research, for without basic research there is no applied research. It's just that one may have to wait for the mathematics to find its application. Kiyoshi Ito's work on stochastic analysis had to wait 50 years before it was implemented in the financial markets, and the Greeks' work on ellipses was put to use in astronomy 1500 years later. So perhaps we have to wait another 50 years for String Theory to be fully understood and put to the test. In the meantime we should keep an open mind and enjoy the new insights it gives to our own discipline.

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